Digital Communication Systems
EES 452

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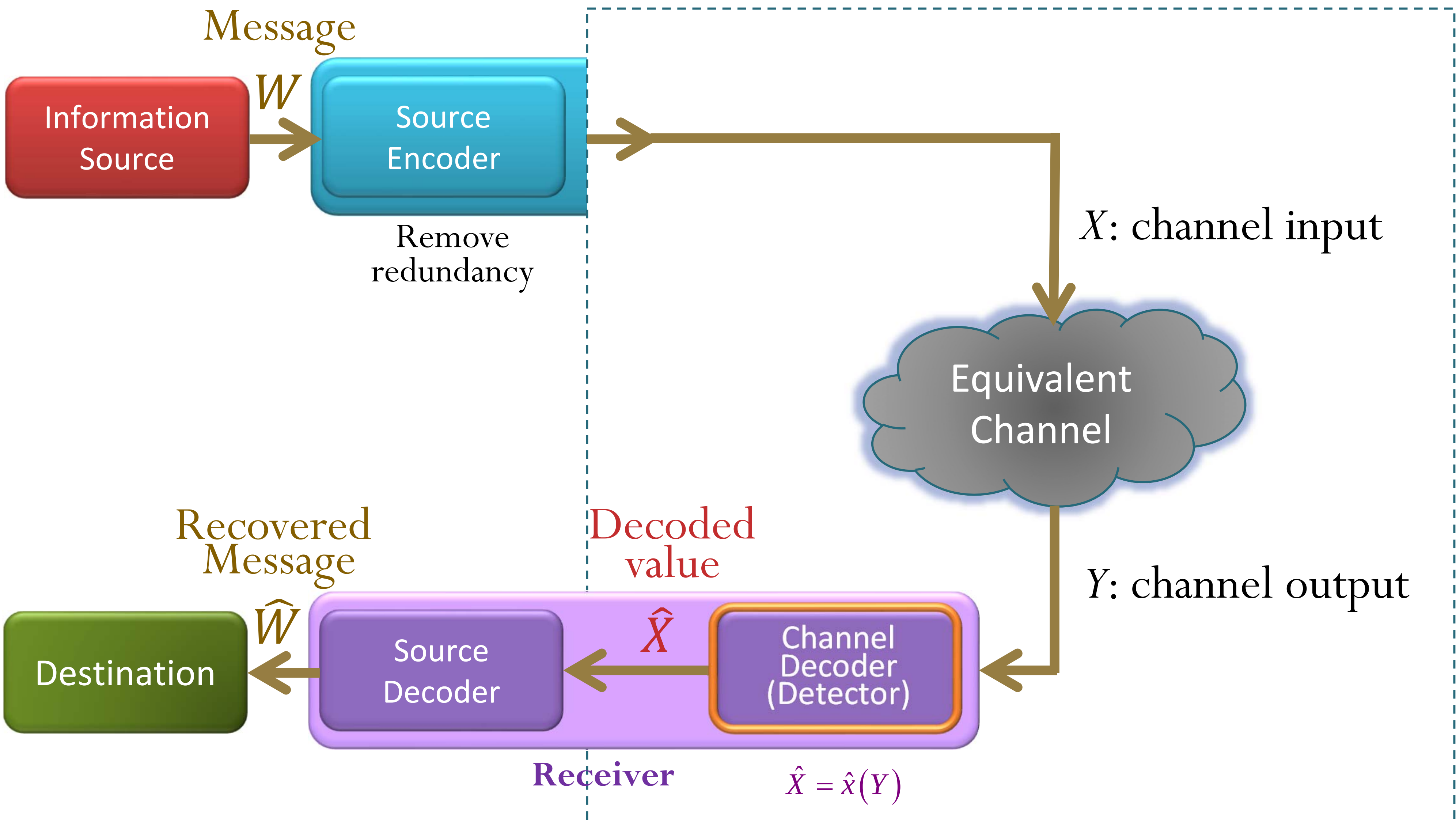
3 An Introduction to Digital Communication Systems Over Discrete Memoryless Channel

3.2 Decoder and $P(\mathcal{E})$

[Figure 11]

System Model for Section 3.2-3.3



Want to minimize $P[\hat{X} \neq X]$

$P(\mathcal{E})$ for Naive Decoder

```
%% Naive Decoder
x_hat = y;
```

```
%% Error Probability
PE_sim = 1-sum(x==x_hat)/n % Error probability from the simulation
```

```
% Calculation of the theoretical error probability
PC = 0;
for k = 1:length(S_X)
    t = S_X(k);
    i = find(S_Y == t);
    if length(i) == 1
        PC = PC+ p_X(k)*Q(k,i);
    end
end
PE_theretical = 1-PC
```

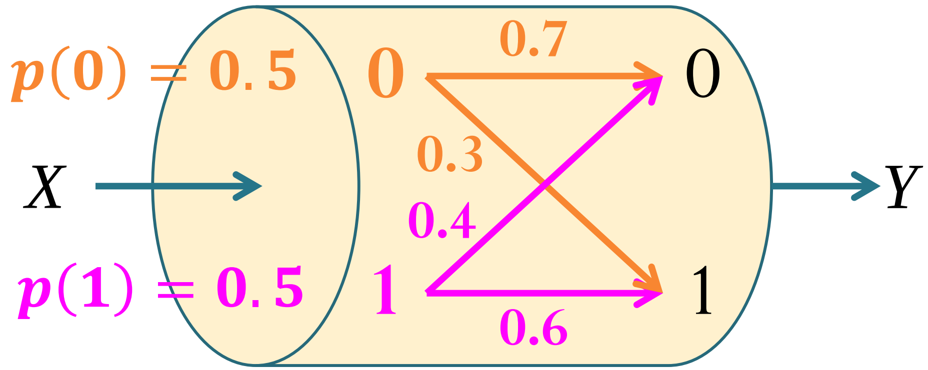
Formula derived in [3.23]

[DMC_Analysis_demo.m]

[Ex. 3.24]: Naive Decoder and BAC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 20;
% Binary Asymmetric Channel (BAC)
% Ex 3.25 in lecture note (11.3 in [Z&T, 2010])
% Channel Input
S_X = [0 1]; S_Y = [0 1];
p_X = [0.5 0.5];
% Channel Characteristics
Q = [0.7 0.3; 0.4 0.6];
```

```
>> BAC_demo
ans =
x:  0 0 0 1 1 0 0 1 0 0 0 0 1 0 0 1 0 1 0 0
ans =
y:  0 0 1 1 0 0 0 1 1 1 0 0 1 0 0 0 0 0 1 0
```




```
p_X =
    0.5000    0.5000
p_X_sim =
    0.7000    0.3000
q =
    0.5500    0.4500
q_sim =
    0.6500    0.3500
Q =
    0.7000    0.3000
    0.4000    0.6000
Q_sim =
    0.7143    0.2857
    0.5000    0.5000
PE_sim =
    0.3500
PE_theretical =
    0.3500
```

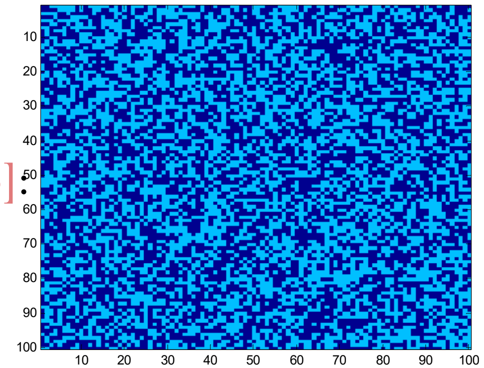
$\frac{7}{20}$ → 0.3500

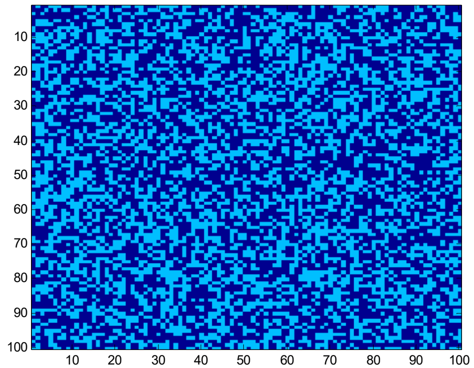
[BAC_demo.m]

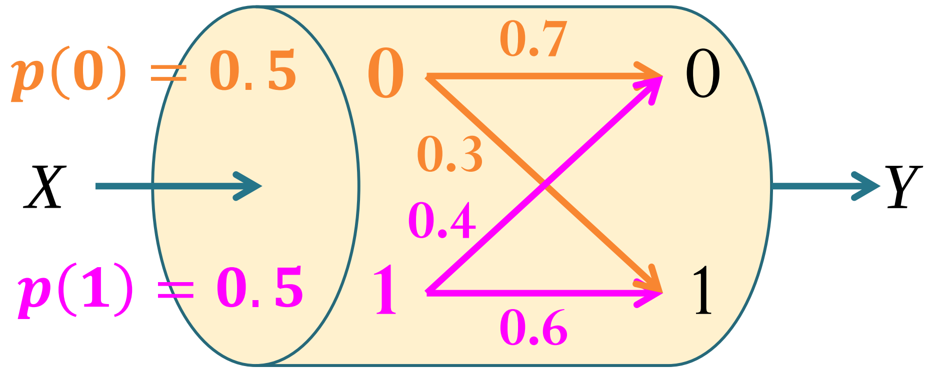
[Ex. 3.24]: Naïve Decoder and BAC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 1e4;
% Binary Assymmetric Channel (BAC)
% Ex 3.24 in lecture note (11.3 in [Z&T, 2010])
% Channel Input
S_X = [0 1]; S_Y = [0 1];
p_X = [0.5 0.5];
% Channel Characteristics
Q = [0.7 0.3; 0.4 0.6];
```

[Ex. 3.16]:





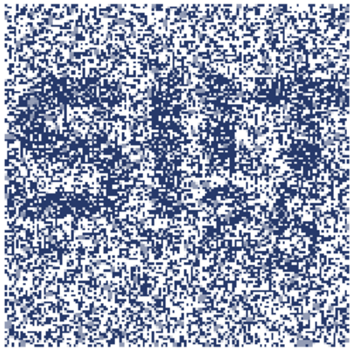


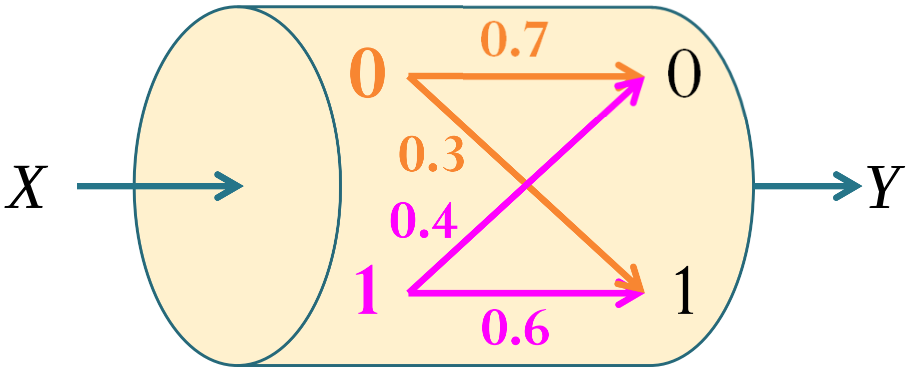

```
p_X =
   0.5000   0.5000
p_X_sim =
   0.5043   0.4957
q =
   0.5500   0.4500
q_sim =
   0.5532   0.4468
Q =
   0.7000   0.3000
   0.4000   0.6000
Q_sim =
   0.7109   0.2891
   0.3928   0.6072
PE_sim =
   0.3405
PE_theretical =
   0.3500
```

[BAC_demo.m]

Naive Decoder and BAC

```
S_X = [0 1]; S_Y = [0 1];
p_X = [0.5 0.5];
% Channel Characteristics
Q = [0.7 0.3; 0.4 0.6];
x_raw = imresize(imread('SIIT_25LOGO.png'),[150 150]);
x_raw = [rgb2gray(x_raw) > 128]; % convert to 0 and 1
x = reshape(x_raw,1,n);
```






```
p_X_sim =
    0.2326    0.7674
q =
    0.4698    0.5302
q_sim =
    0.4672    0.5328
Q =
    0.7000    0.3000
    0.4000    0.6000
Q_sim =
    0.6928    0.3072
    0.3989    0.6011
PE_sim =
    0.3776
PE_theretical =
    0.3767
```

[BAC_demo_image.m]

[Ex. 3.25]: Naive Decoder and DMC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 20;
% General DMC
% Ex. 3.16 in lecture note
% Channel Input
S_X = [0 1]; S_Y = [1 2 3];
p_X = [0.2 0.8];
% Channel Characteristics
Q = [0.5 0.2 0.3; 0.3 0.4 0.3];
```

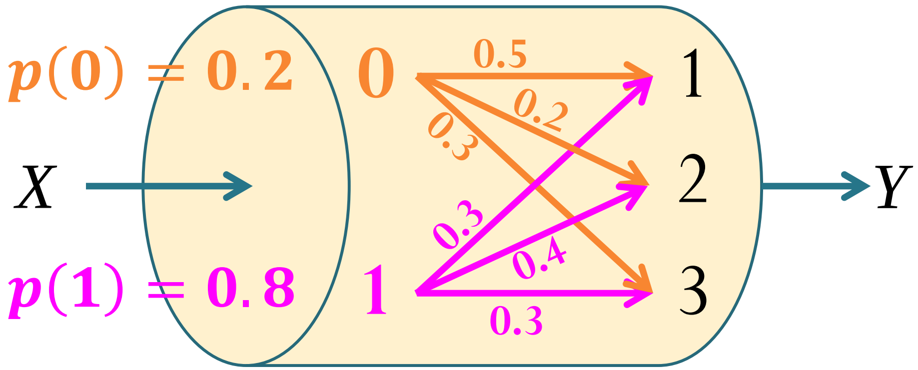
>> DMC_demo [Same samples as in Ex. 3.10]

ans =

x: 1 1 1 1 1 1 1 1 1 1 0 0 1 1 1 0 1 1 0 1

ans =

y: 1 3 2 2 1 2 1 2 2 3 1 1 1 3 1 3 2 3 1 2



$\frac{20-4}{20}$ → 0.7500

p_X =

0.2000 0.8000

p_X_sim =

0.2000 0.8000

q =

0.3400 0.3600 0.3000

q_sim =

0.4000 0.3500 0.2500

Q =

0.5000 0.2000 0.3000

0.3000 0.4000 0.3000

Q_sim =

0.7500 0 0.2500

0.3125 0.4375 0.2500

PE_sim =

0.7500

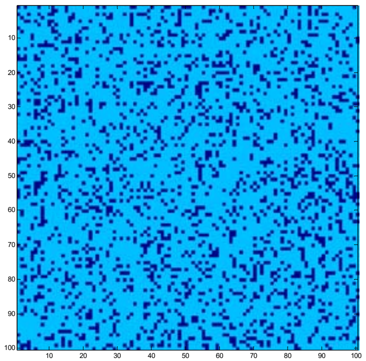
PE_theretical =

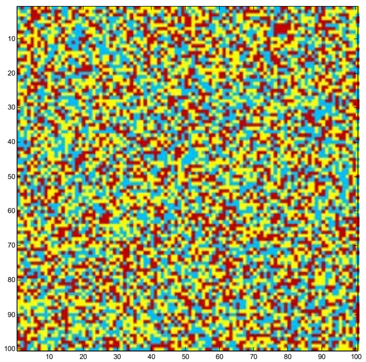
0.7600

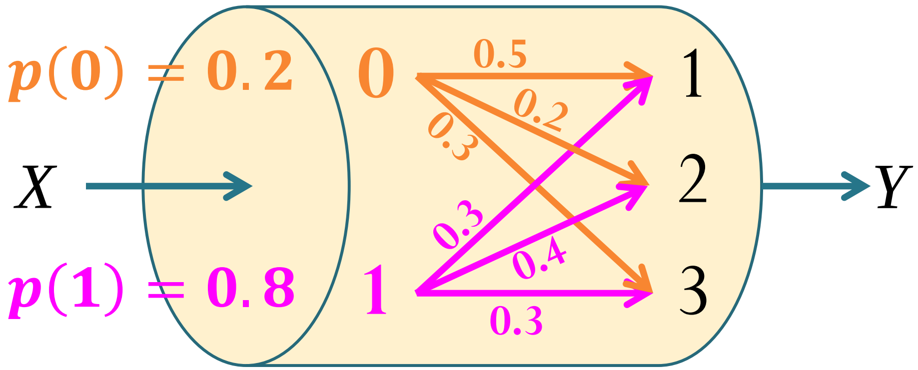
[DMC_demo.m]

[Ex. 3.25]: Naïve Decoder and DMC

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 1e4;
% General DMC
% Ex. 3.16 in lecture note
% Channel Input
S_X = [0 1]; S_Y = [1 2 3];
p_X = [0.2 0.8];
% Channel Characteristics
Q = [0.5 0.2 0.3; 0.3 0.4 0.3];
```







p_X =
0.2000 0.8000

p_X_sim =
0.2011 0.7989

q =
0.3400 0.3600 0.3000

q_sim =
0.3387 0.3607 0.3006

Q =
0.5000 0.2000 0.3000
0.3000 0.4000 0.3000

Q_sim =
0.4943 0.1914 0.3143
0.2995 0.4033 0.2972

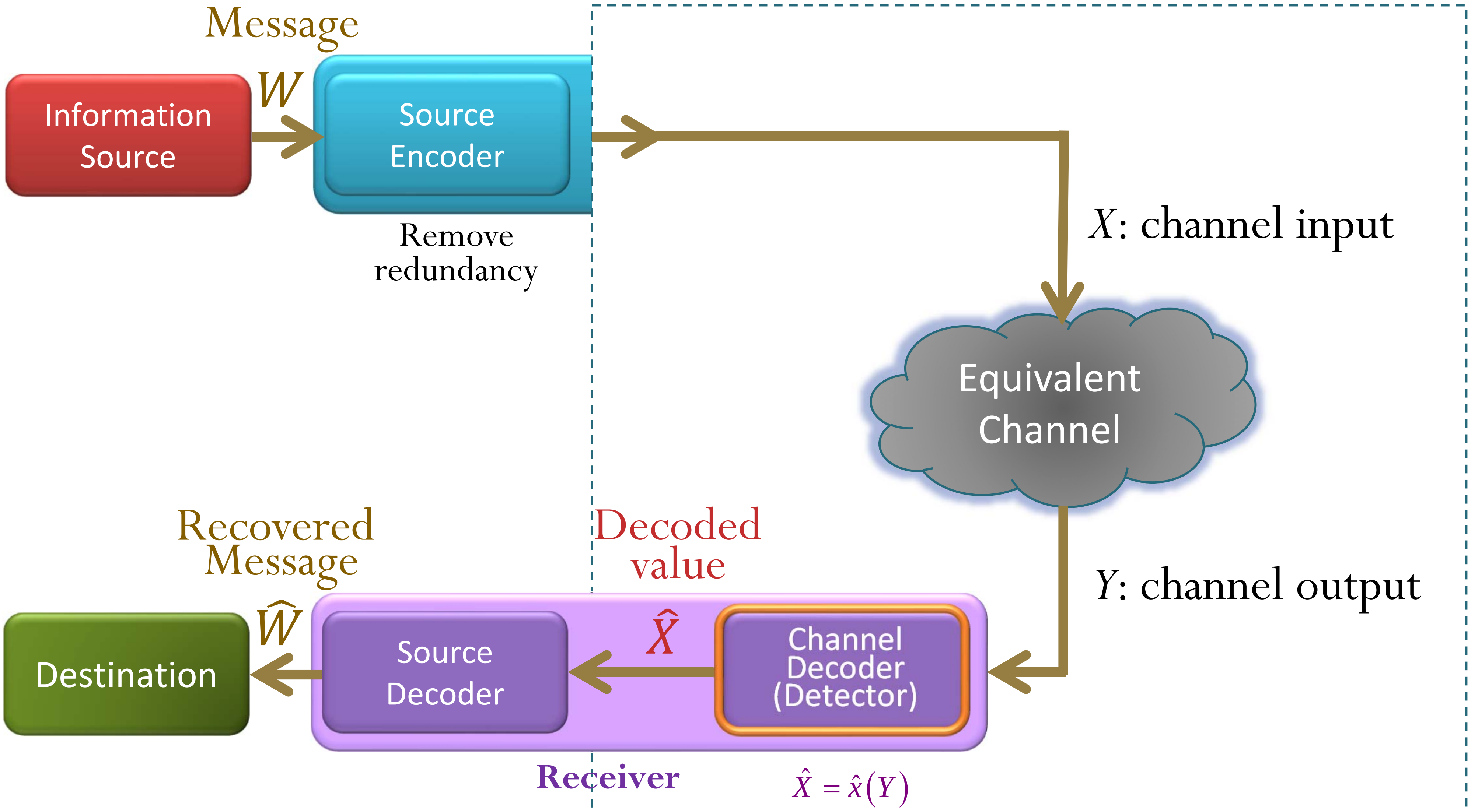
PE_sim =
0.7607

PE_theretical =
0.7600

[DMC_demo.m]

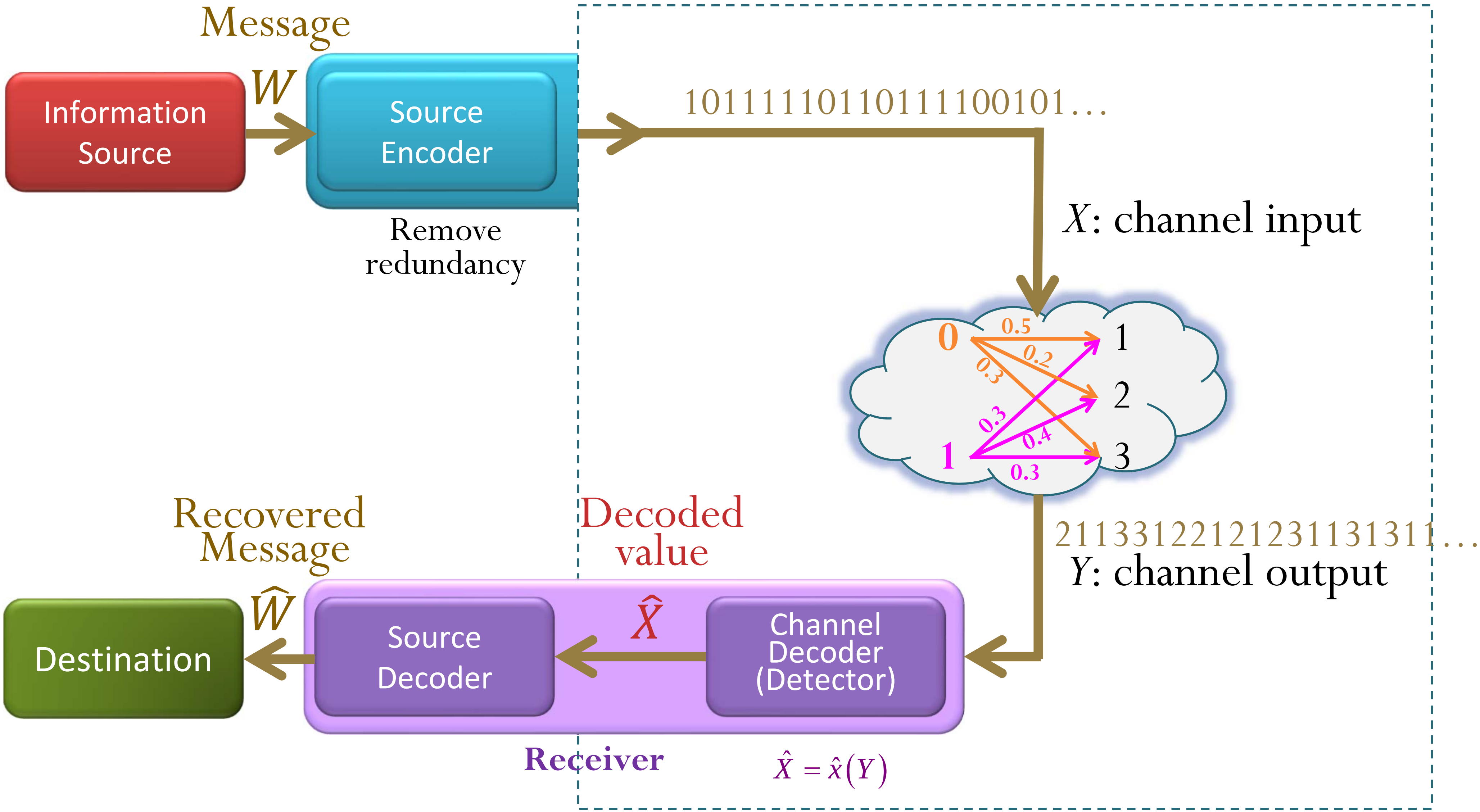
[Figure 11]

System Model for Section 3.2-3.3



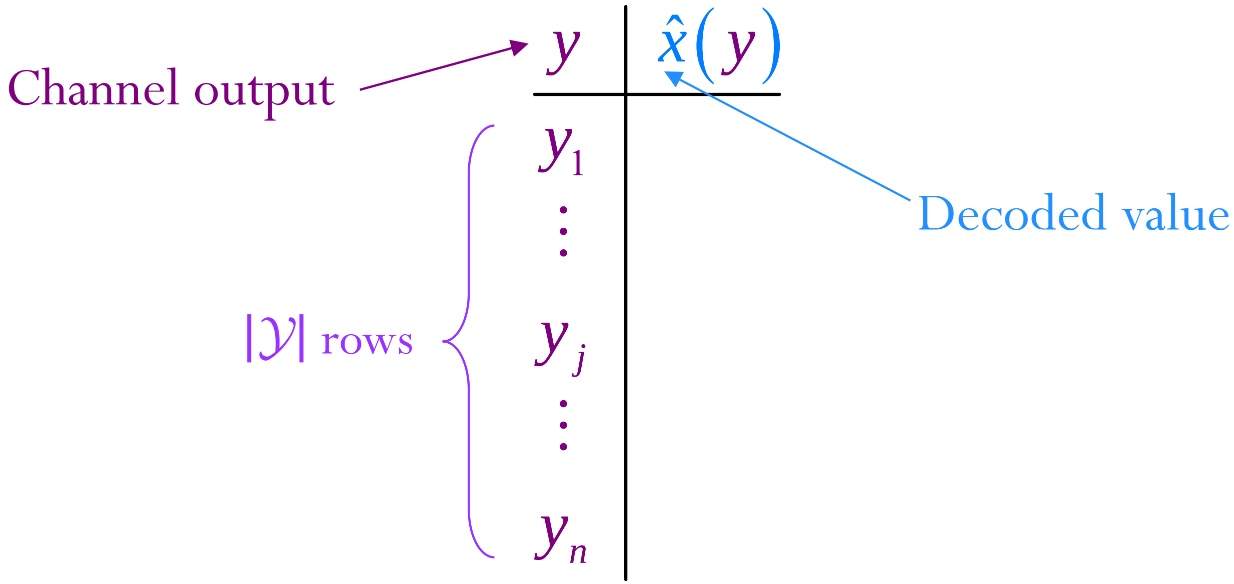
Want to minimize $P[\hat{X} \neq X]$

[Ex. 3.12, 3.17, 3.25]



Defining (Arbitrary) Decoder

Decoding Table



Example 3.25

y	$\hat{x}(y)$
1	1
2	2
3	3

Example 3.27

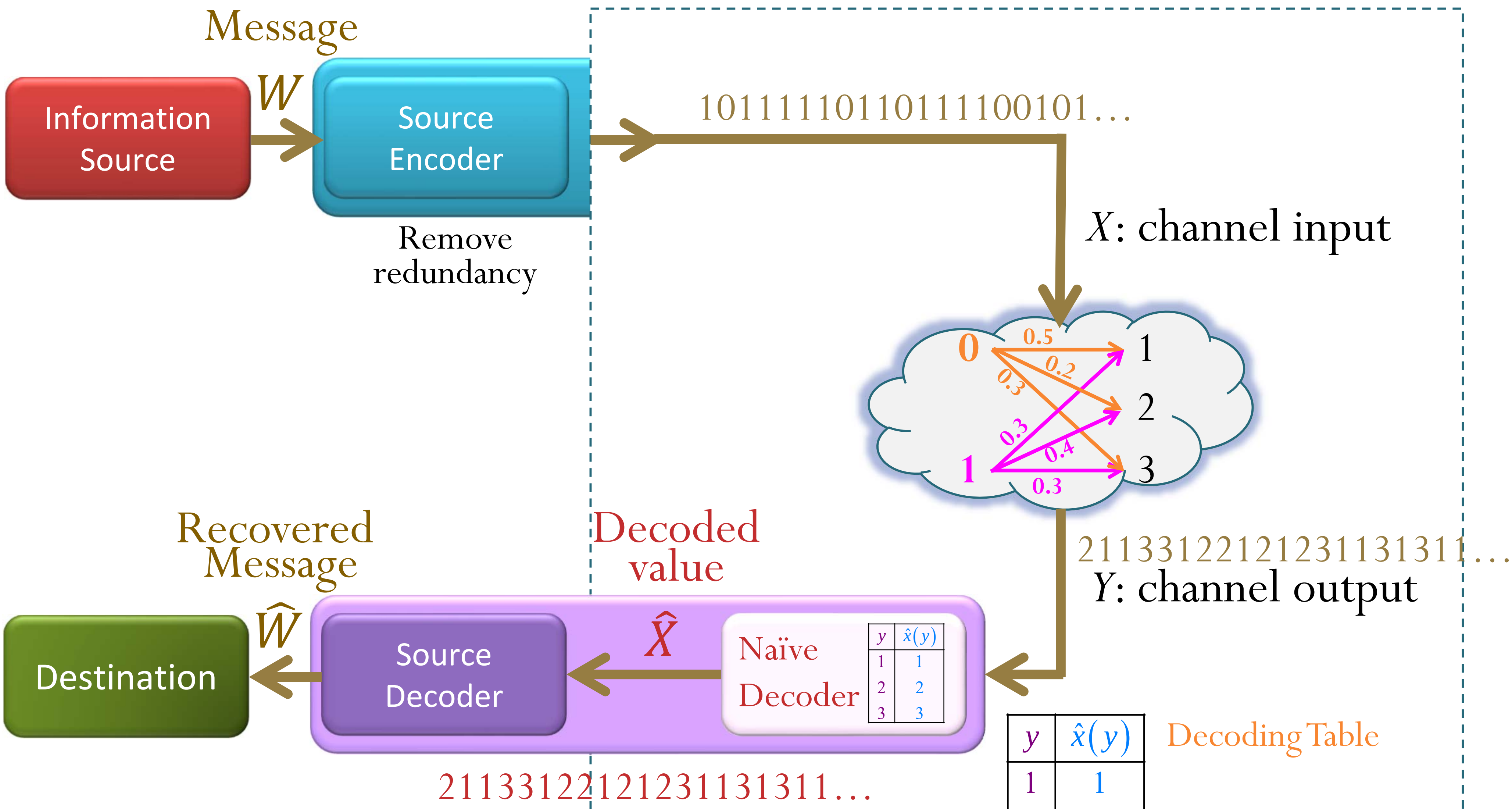
y	$\hat{x}(y)$
1	0
2	1
3	0

Example 3.26

y	$\hat{x}(y)$
1	0
2	1
3	0

[3.29]

[Ex. 3.25]: Naive Decoder

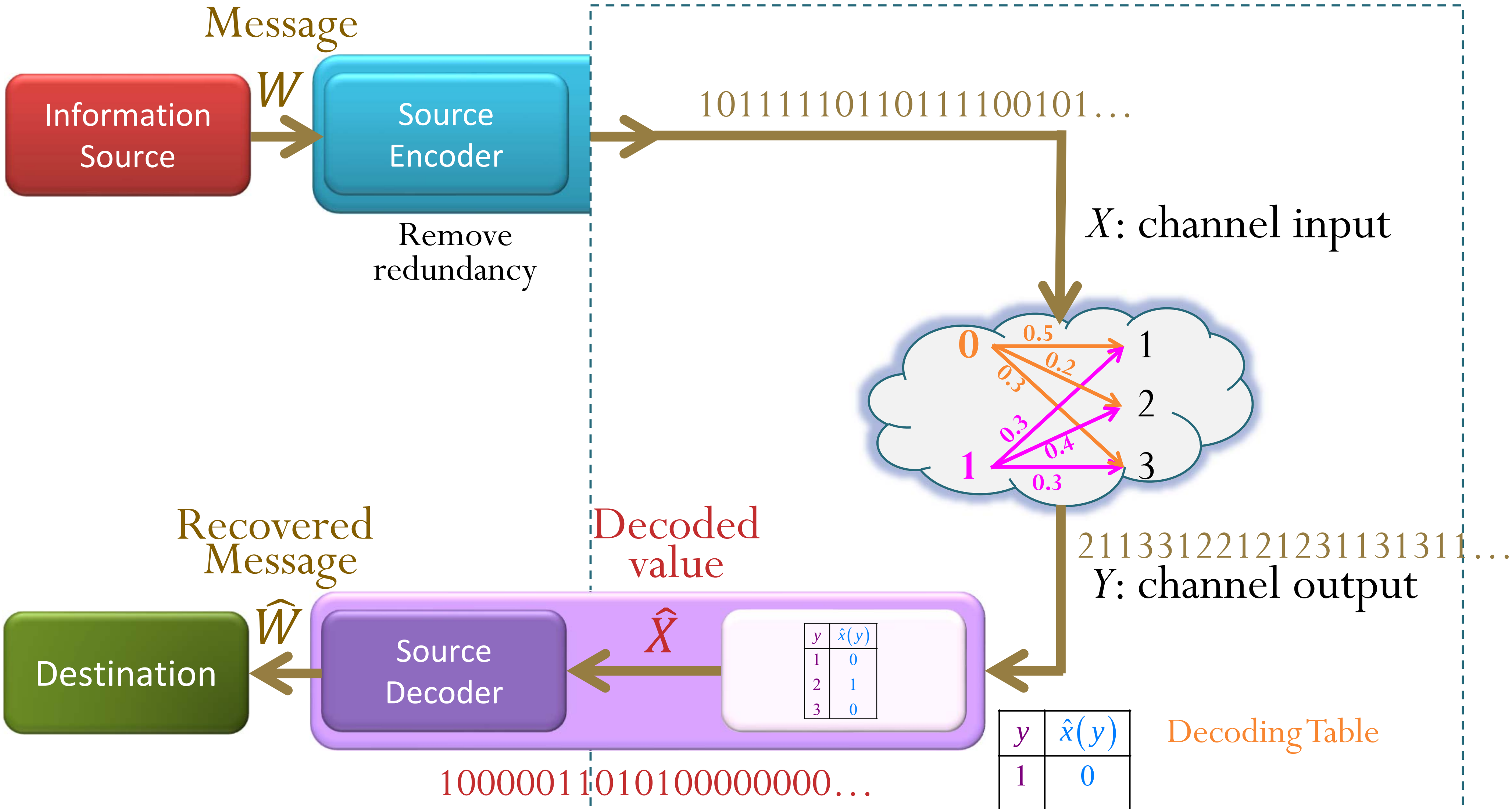


y	$\hat{x}(y)$
1	1
2	2
3	3

Decoding Table

$$P(\mathcal{E}) = P\left[\hat{X} \neq X\right] = 0.76$$

[Ex. 3.26]: DIY Decoder



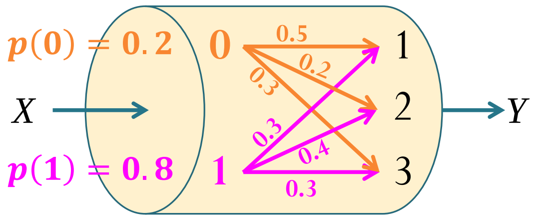
y	$\hat{x}(y)$
1	0
2	1
3	0

Decoding Table

$$P(\mathcal{E}) = P\left[\hat{X} \neq X\right] = 0.52$$

[3.28]

Recipe for finding $P(\mathcal{E})$ of any decoder



y	$\hat{x}(y)$
1	1
2	1
3	0

[Ex. 3.27]

$$\mathbf{Q} = \begin{array}{c|ccc} x \backslash y & 1 & 2 & 3 \\ \hline 0 & 0.5 & 0.2 & 0.3 \\ 1 & 0.3 & 0.4 & 0.3 \end{array} \quad \begin{array}{c} \xrightarrow{\times 0.2} \\ \xrightarrow{\times 0.8} \end{array} \quad \begin{array}{ccc|c} \hat{x}(y): 1 & 1 & 0 & \\ 1 & 2 & 3 & y \diagup x \\ \hline 0.10 & 0.04 & \boxed{0.06} & 0 \\ \boxed{0.24} & \boxed{0.32} & 0.24 & 1 \end{array} = \mathbf{P}$$

- Use the **P** matrix.
  - If unavailable, can be found by scaling each row of the **Q** matrix by its corresponding $p(x)$.
- Write $\hat{x}(y)$ values on top of the y values for the **P** matrix.
- For column y in the **P** matrix, circle the element whose corresponding x value is the same as $\hat{x}(y)$.
- $P(\mathcal{C}) =$ the sum of the circled probabilities.
- $P(\mathcal{E}) = 1 - P(\mathcal{C})$.

Finding $P(\mathcal{E})$

- Once a decoder $\hat{x}(y)$ is defined, we can find its corresponding $P(\mathcal{E})$ easily **from the P matrix**:
  - Write $\hat{x}(y)$ values on top of the y values for the **P** matrix.
  - For each column y in the **P** matrix, circle the element whose corresponding x value is the same as $\hat{x}(y)$.

y	$\hat{x}_{\text{naïve}}(y)$	$\hat{x}_{\text{DIY\_3.26}}(y)$	$\hat{x}_{\text{DIY\_3.27}}(y)$
1	1	0	1
2	2	1	1
3	3	0	0
$P(\mathcal{E})$	0.76	0.52	0.38
	[Ex. 3.25]	[Ex. 3.26]	[Ex. 3.27]

- $P(\mathcal{C})$ = the sum of the circled probabilities.
- $P(\mathcal{E}) = 1 - P(\mathcal{C})$.

$\hat{x}_{\text{naive}}(y)$	1	2	3	$\hat{x}_{\text{DIY\_3.26}}(y)$	0	1	0	$\hat{x}_{\text{DIY\_3.27}}(y)$	1	1	0
$x \setminus y$	1	2	3	$x \setminus y$	1	2	3	$x \setminus y$	1	2	3
0	0.10	0.04	0.06	0	(0.10)	0.04	(0.06)	0	0.10	0.04	(0.06)
1	(0.24)	0.32	0.24	1	0.24	(0.32)	0.24	1	(0.24)	(0.32)	0.24

P =

[Ex. 3.26]: DIY Decoder

```
%% Simulation parameters
% The number of symbols to be transmitted
n = 20;
% General DMC
% Channel Input
S_X = [0 1]; S_Y = [1 2 3];
p_X = [0.2 0.8];
% Channel Characteristics
Q = [0.5 0.2 0.3; 0.3 0.4 0.3];
```

```
>> DMC_decoder_DIY_demo
ans =
X  1 0 1 1 1 1 1 0 1 1 0 1 1 1 1 0 0 1 0 1
ans =
Y  2 1 1 3 3 1 2 2 1 2 1 2 3 1 1 3 1 3 1 1
ans =
X̂  1 0 0 0 0 0 1 1 0 1 0 1 0 0 0 0 0 0 0 0
PE_sim =
   0.5500
PE_thertical =
   0.5200
Elapsed time is 0.081161 seconds.
```

```
%% DIY Decoder
Decoder_Table = [0 1 0]; % The decoded values corresponding to the received Y
```

y	$\hat{x}(y)$
1	0
2	1
3	0

[DMC_decoder_DIY_demo.m]

[Ex. 3.26]: DIY Decoder

```
%% DIY Decoder
Decoder_Table = [0 1 0]; % The decoded values corresponding to the received Y
```

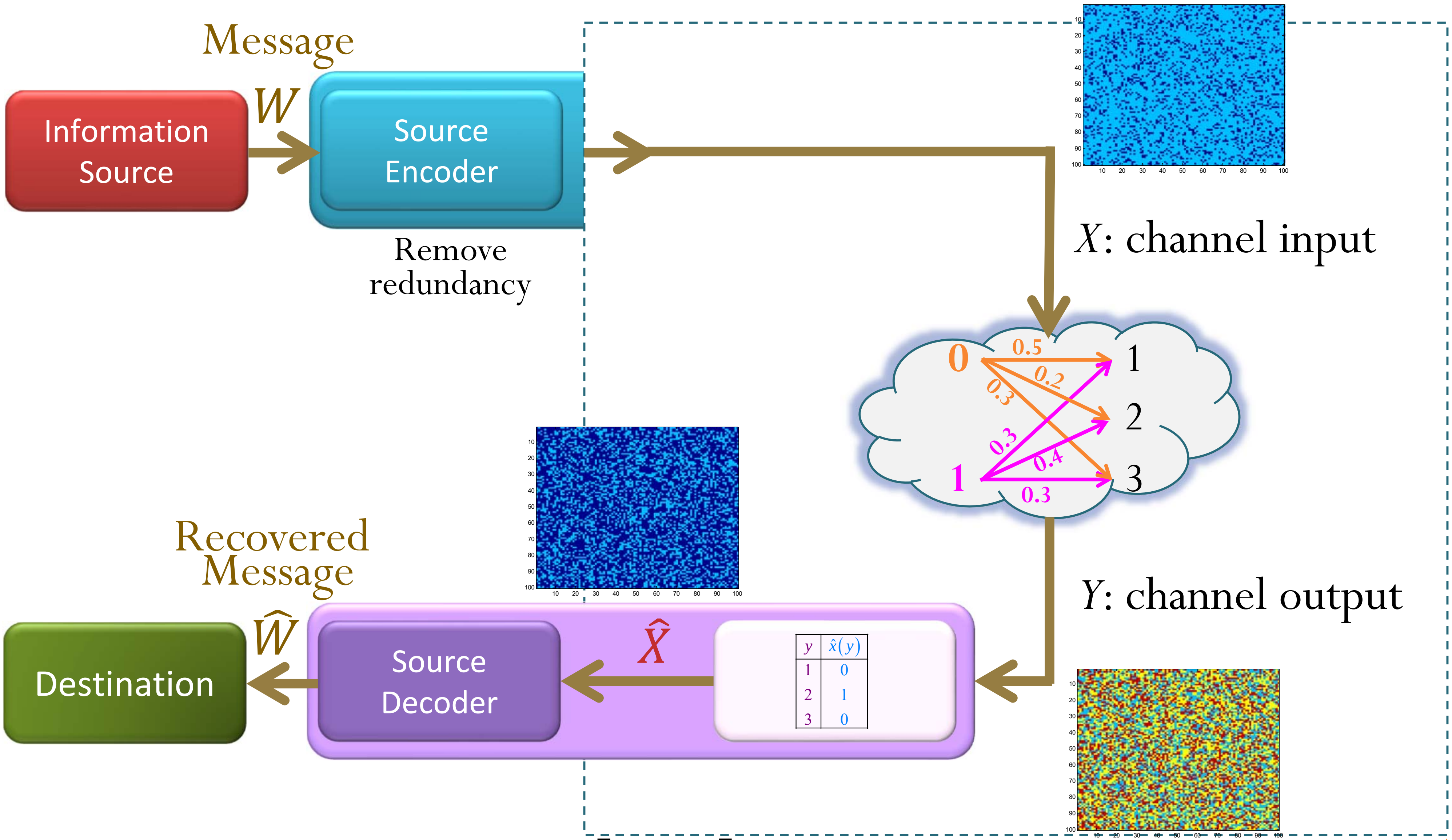
```
% Decode according to the decoder table
x_hat = y; % preallocation
for k = 1:length(S_Y)
    I = (y==S_Y(k));
    x_hat(I) = Decoder_Table(k);
end

PE_sim = 1-sum(x==x_hat)/n % Error probability from the simulation
```

```
% Calculation of the theoretical error probability
PC = 0;
for k = 1:length(S_X)
    I = (Decoder_Table == S_X(k));
    q = Q(k,:);
    PC = PC+ p_X(k)*sum(q(I));
end
PE_theretical = 1-PC
```


[DMC_decoder_DIY_demo.m]

[Ex. 3.26]: DIY Decoder



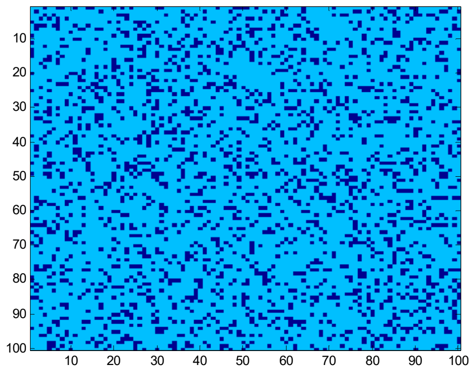
$$P(\mathcal{E}) = P\left[\hat{X} \neq X\right] = 0.52$$

[Ex. 3.26]: DIY Decoder

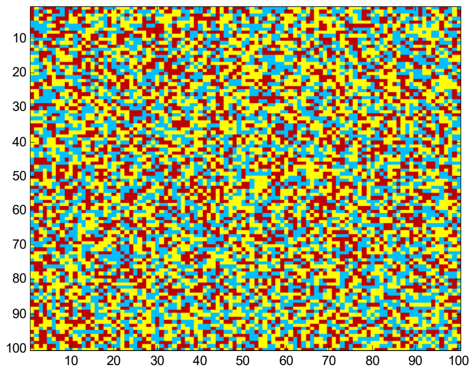
```
%% Simulation parameters
% The number of symbols to be transmitted
n = 1e4;
% General DMC
% Channel Input
S_X = [0 1]; S_Y = [1 2 3];
p_X = [0.2 0.8];
% Channel Characteristics
Q = [0.5 0.2 0.3; 0.3 0.4 0.3];
```

```
>> DMC_decoder_DIY_demo
PE_sim =
   0.5213
PE_theretical =
   0.5200
Elapsed time is 2.154024 seconds.
```

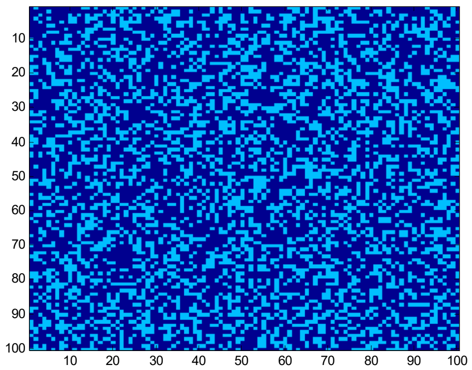
```
%% DIY Decoder
Decoder_Table = [0 1 0]; % The decoded values corresponding to the received Y
```



X



Y



$\hat{X}$

$$P(\mathcal{E}) = P\left[\hat{X} \neq X\right] = 0.52$$

[DMC_decoder_DIY_demo.m]